Twin Primes – An Equivalence

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Abstract

We give a simple characterization of twin primes, allowing one to search for twin primes without working with primes.

For all *a,b* ∈ N we let

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Observe that *n*(*a,a*) = *a*. If *a* 6= *b* then *n*(*a,b*) may or may not lie in N. We define

*M* := {*n*(*a,b*)|*a,b* ∈ N*,a* 6= *b,n*(*a,b*) ∈ N} ⊆ N*.*

Theorem. Let *n* ∈ N. Then 6*n* − 1 and 6*n* + 1 are twin primes if and only if *n* ∈6 *M*.

*Proof.* Let 6*n*−1 and 6*n*+1 be twin primes. Let us assume that *n* ∈ *M*. Then there are *a,b* ∈ N*,a* 6= *b,* such that. In particular, , which implies

(6*n* − 1)(6*n* + 1) = 36*n*2 − 1 = 36*ab* + 6*a* − 6*b* − 1 = (6*a* − 1)(6*b* + 1)*.*

Since 6*n* − 1 and 6*n* + 1 are primes, clearly 6*n* − 1 = 6*a* − 1 and 6*n* + 1 = 6*b* + 1. Hence, *n* = *a* = *b* contradicting *a* 6= *b*. This shows that *n* 6∈ *M*.

Conversely, let 6*n* − 1 and 6*n* + 1 not be twin primes.

Case 1: 6*n* − 1 is not prime. All of its prime factors have the form 6*k*±1, with an odd number of them

being of the form 6*k* − 1. Thus we may write

6*n* − 1 = (6*a* − 1)(6*b* + 1)

for some *a,b* ∈ N (for instance by choosing 6*a* − 1 as one of the prime factors of the form 6*k* − 1 and 6*b* + 1 as the product of all other prime factors). We let

*c* := 6*an* + *a* − *n.*

Then *c > n > b*, so in particular *c* 6= *b.* We obtain

36*n*2 − 1 = (6*n* − 1)(6*n* + 1) = (6*a* − 1)(6*b* + 1)(6*n* + 1) = (6*c* − 1)(6*b* + 1) = 36*bc* + 6*c* − 6*b* − 1

by definition of *c*, yielding

  or *.*

Therefore, *n* ∈ *M*.

Case 2: 6*n* + 1 is not prime. Again, all of its prime factors have the form 6*k* ± 1. But now an even

number (or none) of them has the form 6*k* − 1. Thus we may write

 6*n* + 1 = (6*a* + 1)(6*b* + 1) or 6*n* + 1 = (6*a* − 1)(6*b* − 1)

for some *a,b* ∈ N. (If no prime factor has the form 6*k* − 1 then we may obviously factorize in the first way. Otherwise, we may factorize in the second way by choosing 6*a* − 1 as one of the prime factors of the form 6*k* − 1 and 6*b* − 1 as the product of all other prime factors.)

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Case 2.1: 6*n* + 1 = (6*a* + 1)(6*b* + 1) Then we let

*c* := 6*an* + *n* − *a*

and infer *c > n > b* and *n* ∈ *M* like in case 1.

Case 2.2: 6*n* + 1 = (6*a* − 1)(6*b* − 1) W.l.o.g. *b* ≤ *a*. We let

*c* := 6*an* − *n* − *b*

and again infer *c > n > b* and *n* ∈ *M* like in case 1.

In any case we have shown that *n* ∈ *M*, which completes the proof.